

**A CONSTITUENT QUARK ANTI-QUARK EFFECTIVE LAGRANGIAN
BASED ON THE DUAL SUPERCONDUCTING MODEL OF LONG
DISTANCE QCD**

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ABSTRACT

We review the assumptions leading to the description of long distance QCD by a Lagrangian density expressed in terms of dual potentials. We find the color field distribution surrounding a quark anti-quark pair to first order in their velocities. Using these distributions we eliminate the dual potentials from the Lagrangian density and obtain an effective interaction Lagrangian $L_I(\vec{x}_1, \vec{x}_2; \vec{v}_1, \vec{v}_2)$ depending only upon the quark and anti-quark coordinates and velocities, valid to second order in their velocities. We propose L_I as the Lagrangian describing the long distance interaction between constituent quarks. Elsewhere we have determined the two free parameters in L_I , α_s and the string tension σ , by fitting the 17 known levels of $b\bar{b}$ and $c\bar{c}$ systems. Here we use L_I at the classical level to calculate the leading Regge trajectory. We obtain a trajectory which becomes linear at large M^2 with a slope $\alpha' \simeq .74 \text{ GeV}^{-1}$, and for small M^2 the trajectory bends so that there are no tachyons.

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For a constituent quark mass between 100 and 150 MeV this trajectory passes through the two known Regge recurrences of the π meson. In this paper, for simplicity of presentation, we have treated the quarks as spin-zero particles.

I. INTRODUCTION

In 1964 Gell-Mann and Zweig proposed that the constituents of hadrons are quarks u, d, and s which have masses of order of 300 MeV and which carry color degrees of freedom. Elementary calculations analogous to those used in atomic physics give a good qualitative description of the hadronic spectrum. We know that the forces in an atom are mediated by the electromagnetic field A_μ . One might ask what is the field outside constituent quarks which mediates their interaction? From the point of view of QCD constituent quarks are superpositions of interacting gluons A_μ^a and fundamental spin 1/2 quark fields ψ_α . At short distances where the coupling is weak (asymptotic freedom) one can use the Yang Mills potentials A_μ^a and carry out perturbation calculations. At long distances the coupling becomes strong (anti-screening) and there is no procedure available to use the potentials A_μ^a to calculate the force between quarks.

In this paper we propose that dual potentials C_μ^a mediate the long distance interactions between constituent quarks.[‡] We construct an effective Lagrangian density \mathcal{L}_{eff} to describe the C_μ^a quark interactions, treating the quarks as classical point particles having coordinates $\vec{x}_i(t)$ and velocities $\vec{v}_i(t)$. We solve the resulting classical field equations for C_μ^a in the presence of a quark anti-quark pair to first order in their velocities. Using these solutions we eliminate the dual potentials C_μ^a from \mathcal{L}_{eff} obtaining an effective interaction Lagrangian $L_I(\vec{x}_1, \vec{x}_2; \vec{v}_1, \vec{v}_2) \equiv \int d\vec{x} \mathcal{L}_{\text{eff}}(\vec{x})$ depending upon the quark and anti-quark positions and velocities valid to second order in these velocities. In another paper [1], starting with L_I , we carry out the canonical quantization procedure to construct the Hamiltonian for an interacting quark anti-quark pair and calculate the spectrum of mesons composed of heavy quarks. In this paper we use the effective Lagrangian L_I at the classical level to calculate Regge

[‡]We have previously discussed the motivation for using dual potentials to describe long distance QCD. In order to make this paper self-contained, we present here the arguments and assumptions involved without reference to our earlier work.

trajectories. For the sake of the simplicity of this presentation we do not treat the spin degrees of freedom here. (These have been considered elsewhere [1] [2]). The principal new development is the treatment of the quark's coordinates and velocities as dynamical variables rather than as externally prescribed parameters. This makes possible the construction of the effective Lagrangian L_I depending only upon particle coordinates and velocities. The same procedure can be used to calculate the effective Lagrangian $L_I(\vec{x}_1, \vec{x}_2, \vec{x}_3; \vec{v}_1, \vec{v}_2, \vec{v}_3)$ for the system of three interacting quarks in a baryon.

II. ABELIAN THEORY

We first review how abelian dual potentials C_μ provide an alternate framework to solve Maxwell's equations in a linear relativistic medium having a wave number dependent dielectric constant $\varepsilon(q)$ and magnetic permeability $\mu(q) = 1/\varepsilon(q)$ in the presence of a given charge density $\rho(\vec{x})$ and current density $\vec{j}(\vec{x})$ [3]. We first express ρ and \vec{j} in terms of lines of polarization (Dirac strings) attached to the moving charged particles, i.e. we find a string polarization \vec{P}_s and a string magnetization \vec{M}_s so that

$$\rho = -\vec{\nabla} \cdot \vec{P}_s \quad , \quad \vec{j} = \frac{\partial \vec{P}_s}{\partial t} + \vec{\nabla} \times \vec{M}_s . \quad (2.1)$$

The solution to eq. (2.1) for the case of two particles of charges $e(-e)$ moving along trajectories $\vec{x}_1(t)(\vec{x}_2(t))$ is

$$\vec{P}_s(\vec{x}, t) = e \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y} \delta(\vec{x} - \vec{y}) , \quad (2.2)$$

$$\vec{M}_s(\vec{x}, t) = e \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y} \times \dot{\vec{y}} \delta(\vec{x} - \vec{y}) , \quad (2.3)$$

where the integrals $d\vec{y}$ are along any path $L(t)$ connecting $\vec{x}_1(t)$ and $\vec{x}_2(t)$ and $\dot{\vec{y}}$ is the velocity of the portion of the path between \vec{y} and $\vec{y} + d\vec{y}$. (See Fig. 1). It is readily checked that the string polarization \vec{P}_s and magnetization \vec{M}_s yield eq. (2.1) with

$$\rho = e[\delta(\vec{x} - \vec{x}_1(t)) - \delta(\vec{x} - \vec{x}_2(t))] , \quad (2.4)$$

$$\vec{j} = e[\vec{v}_1(t)\delta(\vec{x} - \vec{x}_1(t)) - \vec{v}_2(t)\delta(\vec{x} - \vec{x}_2(t))]. \quad (2.5)$$

For example taking the divergence of eq. (2.2) yields

$$\vec{\nabla} \cdot \vec{P}_s = e \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y} \cdot \vec{\nabla} \delta(\vec{x} - \vec{y}) = -\rho. \quad (2.6)$$

Inserting expressions (2.1) for ρ and \vec{j} into Gauss' Law, $\vec{\nabla} \cdot \vec{D} = \rho$, and Ampere's Law, $\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$, yields equations for \vec{D} and \vec{H} whose general solution is

$$\vec{D} = -\vec{\nabla} \times \vec{C} - \vec{P}_s, \quad \vec{H} = -\frac{\partial \vec{C}}{\partial t} - \vec{\nabla} C_0 + \vec{M}_s. \quad (2.7)$$

Thus Gauss' Law and Ampere's Law have become kinematical equations whose solution defines dual potentials $C^\mu \equiv (C_0, \vec{C})$.

The homogeneous Maxwell equations, $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, taken together with the constitutive equations, $\vec{B} = \mu(q)\vec{H}$, $\vec{E} = \frac{1}{\epsilon(q)}\vec{D} = \mu(q)\vec{D}$ and eq. (2.7) for \vec{D} and \vec{H} then yield the following dynamical equations for the dual potentials C_μ :

$$\partial^\alpha \mu(\partial_\alpha C_\beta - \partial_\beta C_\alpha) = -\partial^\alpha(\mu G_{\alpha\beta}^s), \quad (2.8)$$

where the string polarization tensor $G_{\alpha\beta}^s$ appearing in the inhomogeneous term in eq. (2.8) has the following components:

$$G_{0k}^s = M_{s,k}, \quad G_{ij}^s = -\epsilon_{ijk}P_{s,k}. \quad (2.9)$$

Eqs. (2.8) provide an alternate form of Maxwell's equations, which are completely equivalent to their usual form expressed in terms of ordinary potentials A_α ,

$$\partial^\alpha \epsilon(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = J_\beta, \quad (2.10)$$

where $J^\beta \equiv (\rho, \vec{j})$. We now show how eqs. (2.2), (2.3), (2.7), (2.8) and (2.9) with $\mu = 1$ give the usual electric and magnetic fields of slowly moving particles.

First consider the case of particles at rest. Then the space component of eq. (2.8), $(\vec{\nabla} \times \vec{E} = 0)$, determining the dual vector potential (denoted $\vec{C}_{\text{DIRAC}} \equiv \vec{C}_D$) takes the form

$$\vec{\nabla} \times (-\vec{\nabla} \times \vec{C}_D) = \vec{\nabla} \times \vec{P}_s. \quad (2.11)$$

Comparing eq. (2.11) with the equation determining the vector potential of a magnetic dipole and using eq. (2.2), we obtain the solution of eq. (2.11):

$$\vec{C}_D = - \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} \frac{ed\vec{y}}{4\pi} \times \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3}. \quad (2.12)$$

Eqs. (2.2), (2.7) and (2.12) then yield the electric field

$$\vec{D} = -\vec{\nabla} \times \vec{C}_D - \vec{P}_s = \vec{D}_{\text{COULOMB}} \equiv \vec{D}_{\text{C}}, \quad (2.13)$$

where

$$\vec{D}_{\text{C}} = \frac{e}{4\pi} \left\{ \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3} - \frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|^3} \right\}. \quad (2.14)$$

The first term $-\vec{\nabla} \times \vec{C}_D$ in eq. (2.13) produces a field analogous to the magnetic field of a line of magnetization, which is the sum of a Coulomb field and field flowing through the string to provide the continuity of the field lines. The second term \vec{P}_s cancels the field through the string leaving the desired pure Coulomb field. See Fig. 2.

For slowly moving charges the time component of eq. (2.8) ($\vec{\nabla} \cdot \vec{B} = 0$), determining the scalar potential (denoted C_{0D}) takes the form:

$$-\nabla^2 C_{0D} = -\vec{\nabla} \cdot \vec{M}_s. \quad (2.15)$$

Comparing eq. (2.15) with the equation for the scalar potential of an electric dipole, and using eq. (2.3), we obtain the solution of eq. (2.15):

$$C_{0D} = \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} \frac{ed\vec{y}}{4\pi} \times \dot{\vec{y}} \cdot \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3}. \quad (2.16)$$

Evaluating the magnetic field \vec{H} from eq. (2.7), and using eqs. (2.3), (2.12) and (2.16), we obtain

$$\vec{H} = -\vec{\nabla} C_{0D} - \frac{\partial \vec{C}_D}{\partial t} + \vec{M}_s = \vec{H}_{\text{BIOT SAVART}} \equiv \vec{H}_{BS}, \quad (2.17)$$

where

$$\vec{H}_{BS} = \frac{e}{4\pi} \left[\frac{\vec{v}_1 \times (\vec{x} - \vec{x}_1)}{|\vec{x} - \vec{x}_1|^3} - \frac{\vec{v}_2 \times (\vec{x} - \vec{x}_2)}{|\vec{x} - \vec{x}_2|^3} \right] \quad (2.18)$$

is the usual Biot Savart magnetic field produced by slowly moving charges.

Eq. (2.1) can be written in the covariant form

$$J_\alpha = -\frac{1}{2} \partial^\beta \epsilon_{\alpha\beta\sigma\lambda} G^{s,\sigma\lambda}, \quad (2.19)$$

which makes manifest the electromagnetic duality connecting eqs. (2.8) and (2.10):

$$C_\mu \longleftrightarrow A_\mu, \quad \mu = \frac{1}{\epsilon} \longleftrightarrow \epsilon, \quad \mu G_{\lambda\mu}^s \longleftrightarrow \frac{1}{2} \epsilon_{\lambda\mu\alpha\beta} G^{s,\alpha\beta}. \quad (2.20)$$

All textbooks on electricity and magnetism could be rewritten using only dual potentials C_μ satisfying eq. (2.8) and the same electromagnetic forces between charged particles would be obtained. The potentials themselves however could be completely different from the conventional A_μ . For example in a dielectric medium where $\epsilon(q) \rightarrow 0$ as $q \rightarrow 0$, corresponding to anti-screening at large distances, the potential A_μ determined from eq. (2.10) would be singular at large distances while the dual potentials C_μ satisfying eq. (2.8) with $\mu = \frac{1}{\epsilon} \rightarrow \infty$ as $q^2 \rightarrow 0$ would be screened at large distance. Use of the A_μ to describe this system would introduce singularities which do not appear in the dual potentials C_μ . These are thus the natural choice of potentials to describe a medium with long range anti-screening.

III. NON ABELIAN THEORY (WITHOUT QUARKS)

The main point of the above discussion is that there is an additional degree of freedom available for the description of gauge theories beyond the usual freedom to make gauge transformations on the potentials A_μ . We will exploit this freedom in our study of non abelian gauge theories. In Yang Mills theory the effective dielectric constant results from the non linear interaction of the vector potentials A_μ leading to asymptotic freedom, i.e. anti-screening. We want to find an effective Lagrangian \mathcal{L}_{eff} to describe the long distance behavior of such a medium in terms of non abelian dual potentials C_μ^a . Mandelstam [4] has given a formal relation between the non abelian dual potentials C_μ^a and the Yang Mills

potentials A_μ^a and 'tHooft [5] has shown that if Yang Mills theory confines quarks then the dual Wilson loop (that defined in terms of dual potentials) satisfies a perimeter law. (This is the gauge invariant non abelian analogue of the correspondence, eq. (2.20), obtained in abelian theory showing that in a medium with anti-screening the dual potentials are non singular at large distances.) To guarantee this smooth long distance behavior of the dual potential we then require

- a) \mathcal{L}_{eff} must generate a mass for the C_μ^a field. This mass provides the shielding of the dual potentials expected from the long distance extrapolation of asymptotic freedom.
- b) \mathcal{L}_{eff} must be invariant under non abelian gauge transformations of the dual potentials. (\mathcal{L}_{eff} must incorporate the fundamental color symmetry of QCD.) We write this gauge transformation in the usual matrix form,

$$C_\mu = \Omega^{-1} C_\mu \Omega + \frac{i}{g} \Omega^{-1} \partial_\mu \Omega, \quad (3.1)$$

where $C_\mu = \sum_{a=1}^8 C_{\mu\frac{1}{2}}^a \lambda_a$, Ω is an $SU(3)$ matrix and $g = \frac{2\pi}{e}$ where e is the Yang Mills coupling constant, i.e. $\alpha_s = \frac{e^2}{4\pi}$. (For $SU(N)$ gauge theory C_μ and Ω are $SU(N)$ matrices, and $\frac{1}{2}\lambda_a$ are the generators $SU(N)$, normalized so that $2 \text{tr } \lambda_a \lambda_b = \delta_{ab}$). The relation $g = \frac{2\pi}{e}$ between the coupling g of the dual potential C_μ^a and the Yang Mills coupling constant e can be obtained from 'tHooft's duality relations [5] between non abelian electric and magnetic flux. We will write down a minimal Lagrangian \mathcal{L}_{eff} satisfying (a) and (b). There are in general an infinite number of terms which can be added to this minimal \mathcal{L}_{eff} compatible with the requirements (a) and (b). However contributions to \mathcal{L}_{eff} involving higher dimension operators should be suppressed at long distances and we retain only the lowest dimension operators. Our third assumption is then

- c) Higher dimension operators not included in \mathcal{L}_{eff} are not quantitatively important for describing long distance phenomena.

To construct \mathcal{L}_{eff} we use the Higgs mechanism to generate the C_μ^a mass and couple C_μ^a to three scalar Higgs fields B_1^a, B_2^a, B_3^a , $a = 1, 2, \dots, N^2 - 1$, in the adjoint representation of

$SU(N)$. The additional effective fields B_i^a , $i = 1, 2, 3$, are needed in order to render \mathcal{L}_{eff} local. Since they couple in a minimal way to the dual potentials they must carry magnetic charge. There can not be any fields in \mathcal{L}_{eff} which transform like the fundamental representation of $SU(N)$ since \mathcal{L}_{eff} must have the same symmetry as Yang Mills theory. (We introduce the coupling to quarks later.) We include in \mathcal{L}_{eff} a Higgs potential $W(B)$ which has a minimum at a non vanishing value of B_i^a . For $SU(3)$, B_i^a are octets defining three $SU(3)$ matrices $B_i = \sum_{a=1}^8 B_i^a \frac{\lambda_a}{2}$, and the vacuum expectation values B_{0i} of B_i have the following structure:

$$B_{01} = B_0 \lambda_7, B_{02} = B_0 (-\lambda_5), B_{03} = B_0 \lambda_2, \quad (3.2)$$

where the value of B_0 is determined by the position of the minimum in the Higgs potential W . The three matrices λ_7 , $-\lambda_5$, and λ_2 transform as a $j = 1$ irreducible representation of an $SU(2)$ subgroup of $SU(3)$ and it is readily shown that there is no $SU(3)$ transformation which leaves all three B_{0i} invariant. The $SU(3)$ gauge symmetry is then completely broken and the eight Goldstone bosons become the longitudinal components of the now massive C_μ . (The ansatz (3.2) is readily extended to $SU(N)$ such that the $SU(N)$ gauge symmetry is completely broken. However in the rest of this paper we restrict ourselves to the case of interest, namely $SU(3)$).

The Lagrangian \mathcal{L}_{eff} takes on the following form:

$$\mathcal{L}_{\text{eff}} = 2 \text{tr} \left\{ -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} + \frac{1}{2} (\mathcal{D}_\mu B_i)^2 \right\} - W(B) + \dots, \quad (3.3)$$

where

$$\mathcal{D}_\mu B_i = \partial_\mu B_i - ig[C_\mu, B_i], \quad (3.4)$$

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - ig[C_\mu, C_\nu]. \quad (3.5)$$

The explicit color structure of W is [6]

$$W = 3\mu^2 \text{tr} B_i^2 + \frac{2}{3} \lambda \{ 3 \text{tr} B_i^2 B_j^2 + (\text{tr} B_i^2)^2 + 2(\text{tr} B_i B_j)^2 \}, \quad (3.6)$$

with $\lambda > 0$ and $\mu^2 < 0$ so that W has a minimum at a non vanishing value of $B_0 = 3\mu/8\sqrt{\lambda}$, which determines the vacuum B_i fields B_{0i} via eq. (3.2)[§]. As indicated above there are an infinite number of terms containing higher dimensional operators which can be added to eq. (3.3), but are not included by assumption (c). The theory then contains three parameters: the dual gauge coupling constant $g = \frac{2\pi}{e}$, the value B_0 of the vacuum expectation value of the Higgs field and λ , the quartic Higgs coupling. The non abelian Lagrangian \mathcal{L}_{eff} , eq. (3.3), resembles the Lagrangian of the abelian Higgs model. We will see that it describes a dual superconductor and provides a concrete realization of the Mandelstam-'tHooft dual superconductor mechanism of confinement [7] [8].

The basic manifestation of the dual superconducting properties of \mathcal{L}_{eff} is that it generates classical equations of motion having static electric flux tube solutions carrying a unit of Z_3 flux along the z axis [9]. These are the non abelian analogues of the Nielsen-Olesen magnetic vortex solutions of the abelian Higgs model [10]. We choose the gauge so that $C_0 = 0$, $\vec{\nabla} \cdot \vec{C} = 0$, and \vec{C} is proportional to \hat{e}_ϕ , the unit vector along the ϕ direction and to the hypercharge matrix $Y = \frac{\lambda_8}{\sqrt{3}}$. The Higgs fields B_i lie along the same directions in color space as their vacuum values B_{0i} , eq. (3.2), i.e.

$$B_1(\vec{x}) = \lambda_7 B(\vec{x}), \quad B_2(\vec{x}) = -\lambda_5 B(\vec{x}), \quad B_3(\vec{x}) = \lambda_2 B'(\vec{x}). \quad (3.7)$$

At large distances ρ from the flux tube $B_i(\vec{x}) \rightarrow B_{0i}$. The non vanishing of B_{0i} produces a color monopole current confining the electric flux. Since \vec{C} is proportional to Y , the non abelian term in eq. (3.5) vanishes. We then have $G_{0i} = 0$, $G_{ij} = \epsilon_{ijk} D_k$, where the color electric field, $\vec{D} = -\vec{\nabla} \times \vec{C}$, is proportional to the hypercharge Y . The color electric flux Φ_S passing through a large circle S perpendicular to the z axis is then

$$\Phi_S = \int_S d\vec{a} \cdot \vec{D} = - \oint d\vec{l} \cdot \vec{C}, \quad (3.8)$$

[§]The expression (3.6) is the minimal counterterm necessary for the renormalizability of \mathcal{L}_{eff} [6].

For simplicity we have not included other operators of the same dimension.

where the line integral in eq. (3.8) runs over the perimeter of S . As $\rho \rightarrow \infty$, $\vec{C} \rightarrow -\frac{\hat{e}_\phi Y}{g\rho}$, and eq. (3.8) gives

$$\Phi_S = \frac{2\pi}{g} Y = eY, \quad (3.9)$$

or

$$e^{ig\Phi_S} = e^{2\pi i Y} = e^{\frac{(2\pi i)}{3}}. \quad (3.10)$$

A continuous deformation in $SU(3)$ of our particular solution into a non abelian configuration will leave unchanged $P \exp(ig \int \vec{C} \cdot d\vec{l}) = e^{\frac{2\pi i}{3}}$. The flux tube then contains one unit of Z_3 flux. The energy per unit length in this flux tube defines the string tension σ . The quantity g^2/λ in \mathcal{L}_{eff} plays the role of a Landau Ginzburg parameter. It was determined by using the trace anomaly relating the difference of the value of energy density at large distances from the flux tube and its value at the center of the flux tube to the gluon condensate G_2 . The result [12] $g^2/\lambda = 5$ is near the border between type I and type II superconductors. We obtain from the numerical integration of the static field equations with $g^2/\lambda = 5$:

$$\sigma = (4.9B_0)^2. \quad (3.11)$$

Having fixed the ratio g^2/λ of the parameters appearing in \mathcal{L}_{eff} we are left with two free parameters to determine the quark anti-quark Lagrangian L_I . These can be taken as σ and $\alpha_s \equiv \frac{e^2}{4\pi} = \frac{\pi}{g^2}$.

IV. THE EFFECTIVE QUARK ANTI-QUARK LAGRANGIAN

To couple C_μ to a quark anti-quark pair we introduce a Dirac string connecting the pair and represent their charge density ρ and current density \vec{j} as polarization charge and polarization current as in eqs. (2.1), (2.2) and (2.3). The only difference is that ρ and \vec{j} are proportional to the hypercharge matrix Y so that the quark (anti-quark) can emit (absorb) the unit of Z_3 flux described by the flux tube solution. The polarization \vec{P}_s and

magnetization \vec{M}_s and hence C_μ are proportional to Y . The non abelian contribution to $G_{\mu\nu}$, (3.5) vanishes and the coupling of dual potentials to quarks proceeds as in the abelian case. All quantities $C_\mu, \vec{D}, \vec{H}, \vec{j}, \rho$ appearing in eqs. (2.1) -(2.7) are automatically proportional to Y , and these equations are otherwise unchanged. We now redefine $G_{\mu\nu}$:

$$G_{0k} \equiv H_k, \quad G_{ij} \equiv \epsilon_{ijk} D_k, \quad (4.1)$$

and eq. (2.7) becomes

$$G_{\mu\nu} = (\partial_\mu C_\nu - \partial_\nu C_\mu + G_{\mu\nu}^s), \quad (4.2)$$

where $G_{\mu\nu}^s$ is given by eq. (2.9). We then obtain the Lagrangian density \mathcal{L}_{eff} describing the coupling of dual gluons to a quark anti-quark pair by inserting eq. (4.2) for $G_{\mu\nu}$ into eq. (3.3). This yields:

$$\mathcal{L}_{\text{eff}} = 2 \text{tr} \left\{ -\frac{1}{4} (\partial_\mu C_\nu - \partial_\nu C_\mu + G_{\mu\nu}^s)^2 + \frac{1}{2} (\mathcal{D}_\mu B_i)^2 \right\} - W. \quad (4.3)$$

We make the same color ansatz, eq. (3.7), for B_i . Varying C_μ in \mathcal{L}_{eff} gives the equations of motion:

$$\partial^\mu (\partial_\mu C_\nu - \partial_\nu C_\mu) = -\partial^\mu G_{\mu\nu}^s + J_\nu^B, \quad (4.4)$$

where

$$J_\nu^B \equiv ig[\mathcal{D}_\nu B_i, B_i] = 6g^2 C_\nu B^2(x). \quad (4.5)$$

We have used the color structure (3.7) of B_i to obtain the second equality in eq. (4.5). All terms in eq. (4.4) are in the Y color direction. The first term on the right hand side of eq. (4.4) is the polarization current $\partial^\mu G_{\mu\nu}^s$ of the moving Dirac string. Taken by itself it would produce color fields satisfying Maxwell's equations (eq. (2.8) with $\mu = 1$). The second term J_ν^B is the monopole current carried by the fields B_i and arises from the non abelian terms in eq. (4.3). It shields the dual potentials so that the color flux emanating from the pair is confined. (In a sense the non abelian coupling of C_μ to B_i in eq. (4.4) plays the role of the magnetic permeability μ in eq. (2.8)). Varying B_i gives the equations for B_i , namely

$$\mathcal{D}_\mu \mathcal{D}^\mu B_i = \frac{1}{2} \frac{\delta W}{\delta B_i}, \quad (4.6)$$

which have components in the λ_2 , λ_5 and λ_7 directions in color space.

The solutions of equations (4.4) and (4.6) depend parametrically upon $\vec{R}(t) = \vec{x}_1(t) - \vec{x}_2(t)$, \vec{v}_1 and \vec{v}_2 . We will see that to first order in \vec{v}_1 and \vec{v}_2 the time dependence of C_μ and B_i appears only via the explicit time dependence of \vec{R} , \vec{v}_1 and \vec{v}_2 , i.e.,

$$C_\mu(\vec{x}, t) = C_\mu(\vec{x}, \vec{R}, \vec{v}_1, \vec{v}_2), \quad B_i(\vec{x}, t) = B_i(\vec{x}, \vec{R}, \vec{v}_1, \vec{v}_2). \quad (4.7)$$

Substituting the first order solutions (4.7) into \mathcal{L}_{eff} and thereby eliminating the field variables C_μ and B_i gives the effective quark anti-quark Lagrangian

$$L_I(\vec{x}_1, \vec{x}_2; \vec{v}_1, \vec{v}_2) = \int d\vec{x} \mathcal{L}_{\text{eff}}(C_\mu, B_i), \quad (4.8)$$

which we will see is valid to second order in the velocities. The effective interaction Lagrangian L_I (4.8) supplemented by the contribution of the quark kinetic energy is our starting point for the constituent quark model. It contains the quark masses, B_0 , λ , and $g = \frac{2\pi}{e}$ as parameters. However, $\lambda = g^2/5 = \frac{\pi}{5\alpha_s}$ and $B_0 = \sqrt{\sigma}/4.9$ (eq. (3.11)). Thus the parameters appearing in L_I are α_s and the string tension σ .

We now outline the procedure used to calculate L_I . First we express \mathcal{L}_{eff} in terms of the color electric and magnetic fields \vec{D} and \vec{H} . Using the fact that $-(\partial_\mu C_\nu - \partial_\nu C_\mu + G_{\mu\nu}^s)^2 = 2(\vec{H}^2 - \vec{D}^2)$, we can write eq. (4.3) as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_2, \quad (4.9)$$

where

$$\mathcal{L}_0 = -2 \text{tr} \left\{ \frac{\vec{D}^2}{2} + \frac{(\vec{D} B_i)^2}{2} \right\} - W, \quad (4.10)$$

and

$$\mathcal{L}_2 = 2 \text{tr} \left\{ \frac{\vec{H}^2}{2} + \frac{(\mathcal{D}_0 B)^2}{2} \right\}. \quad (4.11)$$

All terms in \mathcal{L}_{eff} with time derivatives are contained in \mathcal{L}_2 and for a static configuration with $C_0 = 0$, $\mathcal{L}_{\text{eff}} = \mathcal{L}_0$. The Lagrangian L_I can then be written as

$$L_I = \int d\vec{x} \mathcal{L}_0 + \int d\vec{x} \mathcal{L}_2. \quad (4.12)$$

Next using eqs. (2.9) and (4.5) we write eq. (4.4) in the three dimensional form:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{C}) + \partial_0^2 \vec{C} + \frac{\partial}{\partial t} \vec{\nabla} C_0 = -\vec{\nabla} \times \vec{P}_s + \frac{\partial \vec{M}_s}{\partial t} - 6g^2 B^2 \vec{C}, \quad (4.13)$$

$$-\nabla^2 C_0 - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{C} = -\vec{\nabla} \cdot \vec{M}_s - 6g^2 B^2 C_0. \quad (4.14)$$

For quarks at rest $\vec{v}_1 = \vec{v}_2 = 0$, $C_0 = 0$ and eqs. (4.6) and (4.13) become coupled static non linear equations for \vec{C} and B_i . Eq. (4.13) becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{C}) + 6g^2 B^2 \vec{C} = -\vec{\nabla} \times \vec{P}_s, \quad (4.15)$$

which differs from eq. (2.11) by the presence of the monopole current $6g^2 B^2 \vec{C}$. We make the dependence of the solutions to these static equations on the particle positions explicit, writing

$$\vec{C} = \vec{C}(\vec{x} - \vec{x}_1, \vec{x} - \vec{x}_2), \quad B_i = B_i(\vec{x} - \vec{x}_1, \vec{x} - \vec{x}_2). \quad (4.16)$$

For $g = 0$, eq. (4.15) reduces to eq. (2.11) and $\vec{C} = \vec{C}_D$, eq. (2.12), multiplied by Y . For $g \neq 0$ we solve eqs. (4.6) and (4.15) numerically [11]. At large distances the monopole current screens \vec{C} and \vec{C} has the behavior:

$$\begin{aligned} \vec{C}(\vec{x}) &\longrightarrow \vec{C}_D(\vec{x})Y, \quad \text{as } \vec{x} \rightarrow 0, \\ &\longrightarrow \text{decreases exponentially as } \vec{x} \rightarrow \infty. \end{aligned} \quad (4.17)$$

while

$$B_i(\vec{x}) \rightarrow B_{0i}, \quad \text{as } \vec{x} \rightarrow \infty. \quad (4.18)$$

We insert these static solutions into \mathcal{L}_0 to obtain the static potential $V_0(R)$:

$$V_0(R) = - \int d\vec{x} \mathcal{L}_0(\vec{C}, B_i). \quad (4.19)$$

For $g = 0$, $\vec{D} = \vec{D}_c Y$ and

$$V_0(R) = 2 \text{tr} Y^2 \int d\vec{x} \frac{(\vec{D}_c)^2}{2} = -\frac{4}{3} \frac{\alpha_s}{R}. \quad (4.20)$$

The quark self energies as well as the vacuum energy $W(B_0)$ have been subtracted out in eq. (4.20). For $g \neq 0$ $V_0(R)$ has the behavior

$$\begin{aligned} V_0(R) &\longrightarrow -\frac{4}{3} \frac{\alpha_s}{R}, \quad \text{as } R \rightarrow 0, \\ &\longrightarrow \sigma R, \quad \text{as } R \rightarrow \infty, \end{aligned} \quad (4.21)$$

where σ is the string tension determined by the flux tube solution. The color electric field \vec{D} evolves from a dipole field as $R \rightarrow 0$ to a flux tube distribution for $R \rightarrow \infty$. $V_0(R)$ is approximately the sum of a screened Coulomb potential and a linear potential since the dual potential is screened at distances R greater than the screening length $\left(\frac{1}{6g^2 B_0^2}\right)^{1/2}$. (See eq. (4.15)). For completeness we present an analytic parametrization of $V_0(R)$, eq. (4.19), evaluated with the numerical solutions (4.16) of the static field eqs. (4.15) and (4.6) [1]:

$$V_0(R) = -\frac{4}{3} \frac{\alpha_s}{R} e^{-0.511 \sqrt{\frac{\sigma}{\alpha_s}} R} + \sigma R - 0.646(\alpha_s \sigma)^{1/2}. \quad (4.22)$$

Next consider quarks in motion. To first order in the velocities \vec{v}_1, \vec{v}_2 of the quarks the static field configurations (4.16) follow adiabatically the positions $\vec{x}(t)(\vec{x}_2(t))$ of the quark (anti-quark). This follows from the fact that all terms involving time derivatives in eq. (4.13) for \vec{C} and eq. (4.6) for B_i are second order in \vec{v}_1 and \vec{v}_2 . Thus replacing \vec{x}_1 by $\vec{x}_1(t)$ and \vec{x}_2 by $\vec{x}_2(t)$ in the static solution eq. (4.16), we obtain the solution of these equations to first order in the velocities:

$$\vec{C} = \vec{C}(\vec{x} - \vec{x}_1(t), \vec{x} - \vec{x}_2(t)), \quad B_i = B_i(\vec{x} - \vec{x}_1(t), \vec{x} - \vec{x}_2(t)). \quad (4.23)$$

Since the quantity $\int d\vec{x} \mathcal{L}_0$ generates the static field equations, it is stationary about solutions to these equations. Thus evaluating it using eqs. (4.23) for \vec{C} and B_i gives an expression

valid to second order in the velocities and eq. (4.19) remains unchanged to this order. All the velocity dependence of L_I then appears in \mathcal{L}_2 .

\mathcal{L}_2 depends quadratically upon $\partial_0 \vec{C}$, $\partial_0 B_i$, and C_0 , all of which are first order in \vec{v}_1 and \vec{v}_2 . To find C_0 to this order we solve eq. (4.14), with \vec{C} and B replaced by their static values. Eq. (4.14) then becomes the following linear equation for C_0 :

$$-\nabla^2 C_0 + 6g^2 B^2 C_0 = -\vec{\nabla} \cdot \vec{M}_s. \quad (4.24)$$

Eq. (4.24) differs from eq. (2.15) by the presence of the monopole charge $6g^2 C_0 B^2$. For $g = 0$ eq. (4.24) reduces to eq. (2.15) and $C_0 = C_{0D} Y$. (See eq. (2.12)). For $g \neq 0$ we solve eq. (4.24) numerically and find the solution C_0 , which has the following behavior:

$$\begin{aligned} C_0(\vec{x}) &\longrightarrow C_{0D}(\vec{x})Y \quad \text{as } \vec{x} \rightarrow 0, \\ &\longrightarrow \text{decreases exponentially as } \vec{x} \rightarrow \infty. \end{aligned} \quad (4.25)$$

The time derivatives $\partial_0 B$ and $\partial_0 \vec{C}$ are obtained from the static solutions, attached to the moving particles. (See eqs. (4.23)). For example,

$$\partial_0 B_i = (\vec{v}_1 \cdot \vec{\nabla}_1 + \vec{v}_2 \cdot \vec{\nabla}_2) B_i(\vec{x} - \vec{x}_1(t), \vec{x} - \vec{x}_2(t)). \quad (4.26)$$

Insertion of these expressions for C_0 , $\partial_0 B_i$, and $\partial_0 \vec{C}$ in \mathcal{L}_2 gives the velocity dependent potential V_2 :

$$V_2 = - \int d\vec{x} \mathcal{L}_2(\vec{H}, \mathcal{D}_0 B_i). \quad (4.27)$$

For $g = 0$, $\mathcal{D}_0 B_i = 0$ and $\vec{H} = \vec{H}_{BS} Y$ (see eq. (2.18)). Then

$$V_2 = -2 \text{tr} Y^2 \int d\vec{x} \frac{(\vec{H}_{BS})^2}{2} = \frac{2 \alpha_s}{3 R} \left\{ \vec{v}_1 \cdot \vec{v}_2 + \frac{\vec{v}_1 \cdot \vec{R} \vec{v}_2 \cdot \vec{R}}{R^2} \right\}, \quad (4.28)$$

where the self energies have been subtracted out. For $g \neq 0$ eq. (4.27) gives the following expression for V_2

$$\begin{aligned} V_2 &= \frac{[\vec{R} \times \frac{d\vec{R}}{dt}]^2}{4R^2} V_+(R) + \frac{[\vec{R} \times (\vec{v}_1 + \vec{v}_2)]^2}{4R^2} V_-(R) \\ &+ \frac{\vec{R} \cdot (\vec{v}_1 + \vec{v}_2)]^2}{4R^2} V_{\parallel}(R) - \frac{[\vec{R} \cdot \frac{d\vec{R}}{dt}]^2}{4R^2} \left(\frac{4\alpha_s}{3R} \right), \end{aligned} \quad (4.29)$$

where the ‘‘potentials’’ $V_+(R)$, $V_-(R)$, and $V_{\parallel}(R)$ are moments of the field distributions \vec{C} , C_0 and \vec{B}_i , determined by eq. (4.27). The explicit formulae for the integrals defining these moments are given in reference [1]. Using the numerical solution of eq. (4.24) for C_0 to evaluate the integral for V_+ obtained from eq. (4.27), we find the analytical parametrization [1]:

$$V_+(R) = -\frac{2\alpha_s}{3R} e^{-1.137\sqrt{\frac{\sigma}{\alpha_s}}R} - .208\sigma R + 1.118\sqrt{\alpha_s\sigma}. \quad (4.30)$$

Furthermore in section V we show:

$$V_- = -\frac{1}{2}V_0, \quad V_{\parallel} = -\frac{1}{2}V_0 + \frac{R}{2}\frac{dV_0}{dR}. \quad (4.31)$$

Eqs. (4.29), (4.30), (4.31) and (4.22) yield an analytic parametrization of the velocity dependent potential V_2 in terms of α_s and the string tension σ . Combining the static and velocity dependent contributions, eqs. (4.19) and (4.27), to the effective Lagrangian L_I , we have,

$$L_I = -V_0(R) - V_2. \quad (4.32)$$

For $g = 0$, Eqs. (4.20), (4.28), and (4.32) give,

$$L_I = \frac{4}{3}\frac{\alpha_s}{R} - \frac{2}{3}\frac{\alpha_s}{R} \left\{ \vec{v}_1 \cdot \vec{v}_2 + \frac{\vec{v}_1 \cdot \vec{R} \vec{v}_2 \cdot \vec{R}}{R^2} \right\} \equiv L_D, \quad (4.33)$$

where L_D is the Darwin Lagrangian (multiplied by the color factor $\frac{4}{3} = 2 \text{tr } Y^2$) describing the interaction of electrically charged particles to second order in their velocities [12]. For $g \neq 0$ L_I has the following behavior:

$$L_I \longrightarrow L_D, \quad \text{as } R \rightarrow 0, \quad (4.34)$$

$$L_I \longrightarrow -\sigma R \left\{ 1 - \frac{[\vec{R} \times (\vec{v}_1 + \vec{v}_2)]^2}{8R^2} \right\} + A \frac{(\vec{R} \times \frac{d\vec{R}}{dt})^2}{4R}, \quad \text{as } R \rightarrow \infty, \quad (4.35)$$

where A is a constant determined from the large R behavior of V_+ , i.e.,

$$V_+(R) \longrightarrow -AR, \quad \text{as } R \rightarrow \infty. \quad (4.36)$$

From eq. (4.30) we see that

$$A \approx .208\sigma. \quad (4.37)$$

In obtaining eq. (4.35), we have made use of eqs. (4.29), (4.22), (4.30), and (4.31).

V. DISCUSSION

a) The Lagrangian L_I , eq. (4.32), describes the quark anti-quark interaction at distances R greater than the flux tube radius, $R_{FT} \sim .4fm$. (See Sections VI and V f). At shorter distances this interaction is described by a Lagrangian L_{PT} obtained from perturbation theory. The Lagrangian L_{PT} is essentially the Darwin Lagrangian L_D , eq. (4.33), with α_s replaced by the running coupling constant. Since $L_I \rightarrow L_D$ as $R \rightarrow 0$, one can obtain a Lagrangian applicable at all distances by using eq. (4.32) for $R > R_{FT}$ and using eq. (4.33) for $R < R_{FT}$ with α_s replaced by the running coupling constant. Therefore the elimination of the gauge fields involved in the construction of L_I provides a natural framework in which to connect the short distance perturbative domain described by A_μ^a to the long distance domain described by the C_μ^a . However, in our preliminary fits of energy levels [2], we have been using L_I at all distances. This should overestimate the short distance contributions. Future calculations using L_{PT} at short distances should somewhat modify the values of α_s and σ obtained by fitting data with L_I alone. Comparison of these two calculations will give an indication of the relative importance of long and short distance contributions to energy levels.

b) Bardinelli, Montaldi, and Prosperi [13] (BMP) have given formal expressions for the QCD velocity dependent potentials in terms of generalized Wilson loops extending previous work of Eichten and Feinberg [14] who had written down analogous expressions for spin-spin and spin orbit potentials. Starting with their general formulae and making use of Lorentz invariance, BMP obtained two relations between the various amplitudes appearing in the velocity dependent potential. We now show that relativistic invariance of the Lagrangian

density \mathcal{L}_{eff} yields eqs. (4.31) determining the potentials V_- and V_{\parallel} in terms of the central potential V_0 . Eqs. (4.31) are equivalent to the relations obtained by BMP using generalized Wilson loops. To derive eqs. (4.31) set $\vec{v}_1 = \vec{v}_2 = \vec{v}$ in eq. (4.29) so that the quark and anti-quark move with the same velocity. Then $\frac{d\vec{R}}{dt} = 0$, and eq. (4.29) for V_2 and hence eq. (4.32) for L_I reduces to

$$-L_I = V_0(R) + V_-(R)v^2 + (V_{\parallel}(R) - V_-(R))\frac{(\vec{R} \cdot \vec{v})^2}{R^2}. \quad (5.1)$$

Now consider an inertial frame S' where the quarks are at rest so that $\mathcal{L}'_2 = 0$. Since $\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_2$ is a scalar, $\mathcal{L}_0 + \mathcal{L}_2 = \mathcal{L}'_0$. Then from eq. (4.12),

$$-L_I = -\int d\vec{x}' \mathcal{L}'_0 = -\int d\vec{x}' \sqrt{1-v^2} \mathcal{L}'_0 = \sqrt{1-v^2} V_0(R'), \quad (5.2)$$

where R' is the distance between the quark anti-quark pair in S' . To order v^2 , $R' = R + \frac{1}{2}(\frac{\vec{R} \cdot \vec{v}}{R})^2 R$, and eq. (5.2) becomes

$$-L_I = (1 - \frac{v^2}{2}) \left\{ V_0(R) + \frac{1}{2}(\frac{\vec{R} \cdot \vec{v}}{R})^2 R \frac{dV_0}{dR} \right\}. \quad (5.3)$$

Comparison of eqs. (5.3) and (5.1) then give the relations (4.31).

c) To see the meaning of the remaining terms in V_2 choose $\vec{v}_1 + \vec{v}_2 = 0$ in eq. (4.29). Then

$$V_2 = \frac{V_+(R)}{4R^2} (\vec{R} \times \frac{d\vec{R}}{dt})^2 - \frac{4\alpha_s}{3R} \frac{(\vec{R} \cdot \frac{d\vec{R}}{dt})^2}{4R^2}. \quad (5.4)$$

Note that the coefficient of the radial oscillation $(\vec{R} \cdot \frac{d\vec{R}}{dt})^2$ comes solely from the Biot Savart contribution (4.28). This has a simple physical interpretation. For a pure radial oscillation $\frac{d\vec{R}}{dt} \propto \vec{R}$, i.e. $\vec{R} \times \frac{d\vec{R}}{dt} = 0$, and only the second(Biot Savart) term in eq. (5.4) survives. This is to be expected since confining forces operate primarily perpendicular to the direction of the flux tube and provide no hindrance to motion along the axis of the flux tube.

To isolate the V_+ contribution to eq. (5.4), consider the special case of circular motion. $\frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}$, $\vec{\omega} \cdot \vec{R} = 0$. Then eq. (5.4) becomes

$$V_2 = \frac{1}{4} R^2 V_+(R) \omega^2 = -\frac{1}{2} I(R) \omega^2, \quad (5.5)$$

where

$$I(R) \equiv -\frac{1}{2}R^2V_+(R) \quad (5.6)$$

is the moment of inertia of the rotating quark anti-quark pair. The constant A (see eqs. (4.36) and (4.37)), determines $I(R)$ for large R . The minus sign in eq. (5.5) arises because V_2 defined by eq. (4.27) is the mechanical potential energy rather than the field energy E_{FIELD} . We have calculated E_{FIELD} and also \vec{J}_{FIELD} , the angular momentum of the color field distribution produced by a quark anti-quark pair undergoing circular motion of frequency $\vec{\omega}$. As expected, we find:

$$E_{FIELD} = \frac{1}{2}I(R)\omega^2, \quad \vec{J}_{FIELD} = I(R)\vec{\omega}, \quad (5.7)$$

where $I(R)$ is given by eq. (5.6). Since \vec{J}_{FIELD} includes contributions from both scalar fields B_i and vector fields C_μ , it automatically accounts for the spin of the dual gluon and hence differs essentially from that calculated in phenomenological flux tubes models.

d) A Monte Carlo lattice gauge theory calculation of the QCD velocity dependent potentials starting from the formulae of BMP would provide a direct check of our predictions for V_2 . Such a calculation would complement the Monte Carlo calculations of Huntley and Michael [15] of the spin orbit potentials which gave data consistent with the predictions of \mathcal{L}_{eff} augmented by spin dependent terms [2]. These spin dependent potentials like V_2 involved only α_s and σ as parameters.

e) Starting from L_I augmented by the spin dependent terms and non-relativistic quark kinetic energy terms, we have constructed the Hamiltonian H for a heavy quark anti-quark system by the canonical procedure. We then determined the parameters α_s , σ , m_c and m_b by finding a best fit to the 17 known levels of $c\bar{c}$ and $b\bar{b}$ systems. The best fit of the parameters were [1]:

$$(i) \alpha_s = .3515$$

$$(ii) \sigma = .214 \text{ GeV}^2$$

(iii) $m_c = 1.317 \text{ GeV}$

(iv) $m_b = 4.747 \text{ GeV}$

Using $g^2/\lambda = 5$ and eq. (3.11) gives the values $\lambda = 1.788$ and $B_0 = 94.4 \text{ MeV}$ for the parameters appearing directly in \mathcal{L}_{eff} . These parameters then give a value $M^2 \equiv 6g^2B_0^2 = (680 \text{ MeV})^2$ for the dual gluon mass squared, (see eq. (4.15)), and a value $M_B = 477 \text{ MeV}$ for the lightest Higgs meson. This latter mass determines the order of magnitude of the the flux tube radius. (See the following section).

f) Although the main purpose of this paper was the construction of an effective quark Lagrangian which contained no explicit reference to the underlying effective local fields, as noted in (c) one can also use \mathcal{L}_{eff} to calculate local quantities such as the energy momentum tensor $T_{\mu\nu}(x)$. We have calculated the energy density $T_{00}(x)$ surrounding a static quark anti-quark pair separated by a distance R . The energy density on the median plane $z = 0$ falls off exponentially with the distance ρ from the flux tube axis, which is compatible with the results of the calculation of analogous quantities in lattice gauge theory [16]. Although the correspondence between local quantities calculated from Monte Carlo data is not completely clear and the comparison is to some extent qualitative, it provides a check on the physical picture underlying L_I . Such tests are not even in principle possible for purely phenomenological quark anti-quark potentials.

The Monte Carlo calculations [16] were carried out in $SU(2)$ gauge theory for an interquark spacing $R \sim 1.5 \text{ fm}$ and gave a root mean squared flux tube radius $R_{FT} \sim .2 \text{ fm}$. The energy distribution calculated from \mathcal{L}_{eff} broadens as a function of R . For $R = .7 \text{ fm}$, $R_{FT} = .3 \text{ fm}$, for $R = 1.3 \text{ fm}$, $R_{FT} = .4 \text{ fm}$, and for $R = 2.6 \text{ fm}$, $R_{FT} = .45 \text{ fm}$, at which point the “flux tube” is close to its maximum width.

g) S. Maedan and T. Suzuki [17] on the basis of somewhat different reasoning have also proposed a Lagrangian describing long distance QCD in terms of dual potentials. The classical equations of motion derived from their Lagrangian do not differ substantially from

the equations obtained from \mathcal{L}_{eff} in the gauge we have chosen. Kamizawa, Matsubara, Shiba and Suzuki [18] have solved the static equations of their theory in the presence of three static quarks and have calculated the static potential $V_0(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ for three quarks in a baryon. The essential complication of the baryon problem is that the source term $\partial^\mu G_{\mu\nu}^s$ in eq. (4.4) involves a string connecting the three quarks. Starting with the static solution to eqs. (4.4) and (4.6) one can solve these equations to first order in the velocity and insert the result into \mathcal{L}_2 , eq. (4.11). Carrying out this procedure would yield a unique velocity dependent potential $V_2 = -\int \mathcal{L}_2 d\vec{x}$ and an effective interaction Lagrangian $L_I(\vec{x}_1, \vec{x}_2, \vec{x}_3; \vec{v}_1, \vec{v}_2, \vec{v}_3)$ for three quarks inside a baryon.

VI. REGGE TRAJECTORIES

To treat mesons containing light quarks and in particular to find Regge trajectories for particles composed only of light quarks it is essential to use relativistic kinematics. The Lagrangian describing a quark anti-quark pair of masses m_1 and m_2 respectively is

$$L = -m_1\sqrt{1-v_1^2} - m_2\sqrt{1-v_2^2} - V_0 - V_2. \quad (5.8)$$

The Lagrangian (5.8) accounts for the kinematic effects of relativistic velocities but cannot account for inelastic processes. We saw that it was possible to define a velocity dependent potential V_2 only to second order in the velocity. As in electrodynamics, the whole concept of a potential dependent only upon particle positions and coordinates has meaning only to order v^2 because of the appearance of radiation to order $(v^2)^2$. We know for relativistic velocities $v^2 \sim 1$ that it is essential to use the relativistic form of the kinetic energy. It makes sense to use eq. (5.8) for relativistic velocities provided the energies are sufficiently low so that inelastic processes are not important. Radiation occurs only for frequencies ω which are greater than the masses of the dual gluon or Higgs field. These masses are of the order of 500 MeV. (See Sec. V e). The inverse of these masses determine R_{FT} , the radius of the flux tube. Hence there is no radiation for frequencies $\omega < 1/R_{FT}$. On the other hand for quarks separated by a distance R , $\omega R < 1$. Therefore at distances $R > R_{FT}$, $\omega < \frac{1}{R_{FT}}$ and

there is no radiation. The dominant relativistic effects are then kinematic and are included in the Lagrangian (5.8).

To calculate the energy of states having large angular momentum, we can use classical dynamics. We find the leading Regge trajectory, i.e. the energy and angular momentum of those states which have the largest angular momentum for a given energy, by considering circular orbits so that

$$\vec{v}_1 \cdot \vec{R} = \vec{v}_2 \cdot \vec{R} = 0.$$

Then the Lagrangian L , eqs. (5.8), (4.32), and (4.29) reduces to

$$L = -m_1 \sqrt{1 - v_1^2} - m_2 \sqrt{1 - v_2^2} - V_0 - \frac{V_+}{4} (\vec{v}_1 - \vec{v}_2)^2 - \frac{V_-}{4} (\vec{v}_1 + \vec{v}_2)^2. \quad (5.9)$$

From eq. (5.9) we calculate the canonical momentum at $\vec{p}_i = \frac{\partial L}{\partial \vec{v}_i}$, $i = 1, 2$, the energy $E = \vec{p}_1 \cdot \vec{v}_1 + \vec{p}_2 \cdot \vec{v}_2 - L$ and the angular momentum $\vec{J} = \vec{x}_1 \times \vec{p}_1 + \vec{x}_2 \times \vec{p}_2$. We find,

$$E = \frac{m_1}{\sqrt{1 - v_1^2}} + \frac{m_2}{\sqrt{1 - v_2^2}} + V_0 - V_+ \frac{(\vec{v}_1 - \vec{v}_2)^2}{4} - V_- \frac{(\vec{v}_1 + \vec{v}_2)^2}{4}, \quad (5.10)$$

and

$$\vec{J} = \frac{\vec{R}}{2} \times \left[\frac{m_1 \vec{v}_1}{\sqrt{1 - v_1^2}} - \frac{m_2 \vec{v}_2}{\sqrt{1 - v_2^2}} - V_+ (\vec{v}_1 - \vec{v}_2) \right]. \quad (5.11)$$

In obtaining eq. (5.11) we have made use of the fact that we are working in the center of mass frame $\vec{p}_1 = -\vec{p}_2$.

Using Hamilton's principle, the Lagrangian (5.9), and the fact that we have circular orbits so that $\vec{v}_1 \cdot \dot{\vec{v}}_1 = \vec{v}_2 \cdot \dot{\vec{v}}_2 = 0$, we obtain the equation of motion

$$\begin{aligned} \frac{m_1 \dot{\vec{v}}_1}{\sqrt{1 - v_1^2}} - \frac{(V_+ + V_-)}{2} \dot{v}_1 - \frac{(V_- - V_+)}{2} \dot{v}_2 = \\ -\frac{\vec{R}}{R} \left[\frac{\partial V_0}{\partial R} + \frac{(\vec{v}_1 - \vec{v}_2)^2}{4} \frac{\partial V_+}{\partial R} + \frac{(\vec{v}_1 + \vec{v}_2)^2}{4} \frac{\partial V_-}{\partial R} \right], \end{aligned} \quad (5.12)$$

along with a corresponding equation obtained by replacing m_1 by m_2 and interchanging \vec{v}_1 and \vec{v}_2 in eq. (5.12). For circular motion of frequency $\vec{\omega}$, $\vec{v}_1 = \vec{\omega} \times \vec{R}_1$ and $\vec{v}_2 = \vec{\omega} \times \vec{R}_2$ where $\vec{R}_1 = R_1 \hat{R}$, $\vec{R}_2 = -R_2 \hat{R}$, and $R_1 + R_2 = R$. We can then write eq. (5.12) in the form

$$\begin{aligned} \omega^2 \left[\frac{m_1 R_1}{\sqrt{1 - \omega^2 R_1^2}} - \frac{(V_+ + V_-) R_1}{2} + \frac{(V_- - V_+) R_2}{2} \right] = \\ \frac{\partial V_0}{\partial R} + \frac{\partial V_+}{\partial R} \left(\frac{\omega^2 R^2}{4} \right) + \omega^2 \frac{\partial V_-}{\partial R} \frac{(R_1 - R_2)^2}{4}, \end{aligned} \quad (5.13)$$

along with the corresponding equation obtained from eq. (5.13) by replacing m_1 by m_2 and interchanging R_1 and R_2 . Eqs. (5.13) determine R_1 and R_2 in terms of ω . Substituting these expressions for R_1 and R_2 into eqs. (5.10) and (5.11) give \vec{J} and E as functions of ω , which upon elimination of R yield the leading Regge trajectory.

For the equal mass case, $m_1 = m_2 = m$, $R_1 = R_2 = \frac{R}{2}$, and eqs. (5.13), (5.10) and (5.11) reduce to

$$\frac{m}{\sqrt{1 - \frac{\omega^2 R^2}{4}}} = \frac{2}{\omega^2 R^2} R \frac{\partial V_0}{\partial R} + V_+ + \frac{\partial V_+}{\partial R} \frac{R}{2}, \quad (5.14)$$

$$E = \frac{2m}{\sqrt{1 - \frac{\omega^2 R^2}{4}}} + V_0 - V_+ \frac{R^2 \omega^2}{4}, \quad (5.15)$$

$$\vec{J} = \frac{R^2 \vec{\omega}}{2} \left[\frac{m}{\sqrt{1 - \frac{\omega^2 R^2}{4}}} - V_+ \right]. \quad (5.16)$$

The Regge trajectory J as a function of E^2 obtained from these equations is plotted in Fig. (3) for the cases of $m = 100$ MeV and $m = 300$ MeV.

We note that for a constituent quark mass of 100 MeV, this trajectory passes through the two known Regge recurrences of the π meson. For any quark mass the trajectory curves so that there are no particles with E^2 negative. For large E^2 the trajectory becomes linear with a slope α' which is determined by the behavior of V_0 and V_+ at large R , eqs. (4.21) and (4.36). For large R , $\frac{\omega^2 R^2}{4} \sim 1$ and eq. (5.14) becomes

$$\begin{aligned} \frac{m}{\sqrt{1 - \frac{\omega^2 R^2}{4}}} = \frac{1}{2} \sigma R - A R - \frac{A R}{2} = \frac{1}{2} (\sigma - 3A) R. \\ R \rightarrow \infty \end{aligned} \quad (5.17)$$

Note that the presence of the velocity dependent potential V_+ tends to destabilize the system, and there are solutions to eq. (5.17) for ω only for $A < \sigma/3$. This condition is satisfied by the result (4.37) of our numerical calculation.

Inserting the equilibrium condition eq. (5.17) into eqs. (5.15) and (5.16) yields the following expressions for E and \vec{J} valid at large R

$$E = 2R(\sigma - A), \quad \vec{J} = \hat{\omega} \frac{R^2}{2}(\sigma - A). \quad (5.18)$$

Eq. (5.18) then yields a linear Regge trajectory $|\vec{J}| = \alpha' E^2$ with

$$\alpha' = \frac{1}{8\sigma(1 - \frac{A}{\sigma})} \approx \frac{1}{\sigma(6.34)}. \quad (5.19)$$

Note the relativistic Salpeter Equation (with a central potential V_0) gives, for large l , linear Regge trajectories with $\alpha' = \frac{1}{8\sigma}$. This agrees with our classical result for $V_+ = 0$. Including the velocity dependent potential then increases the slope of the Regge trajectories. Using the value $\sigma = .214 \text{ GeV}^2$, (see Sec. V e), we obtain

$$\alpha' = \frac{1}{1.36} \text{ GeV}^{-2} = .74 \text{ GeV}^{-1}. \quad (5.20)$$

In collaboration with Lewis Fulcher we are now in the process of calculating the energy levels of mesons containing light quarks using a Hamiltonian obtained from the Lagrangian (5.7). The velocity dependent terms in the Hamiltonian are uniquely determined from the canonical procedure applied to V_2 (4.29). Since these velocity dependent terms will be more important in mesons containing light quarks, the comparison with experiment will be more sensitive to our prediction for V_2 . Furthermore, since the relevant distance scales will be larger than in mesons containing only heavy quarks, the long distance part of the potential will give a corresponding larger contribution to the energy levels. For both these reasons we feel that these calculations of energy levels of mesons containing light quarks will provide a sensitive test of the theory. The only new parameters are the constituent masses of the light quarks.

VI. SUMMARY AND COMMENTS

The original quark model hypothesized the existence of constituent quarks. It left the nature of their interaction an open question. We have argued that dual potentials are the natural variables describing long distance QCD and have written down an effective Lagrangian \mathcal{L}_{eff} determining their interaction with quarks. Since dual potentials describe only long distance QCD they cannot have a local interaction with the fundamental quarks of QCD. We therefore assume that the dual potentials in \mathcal{L}_{eff} interact with constituent quarks. The electromagnetic interaction can be used to define electrons and positrons as the source of photons. Similarly the interaction mediated by dual potentials can be used to define constituent quarks as the source of dual gluons. Solving the field equations for these dual potentials in the presence of a quark anti-quark pair leads to an effective Lagrangian for constituent quarks inside mesons, containing α_s and σ as parameters. Physically this Lagrangian is the non abelian analogue of the effective Lagrangian describing the interaction of a monopole anti-monopole pair in a superconductor. It thus provides a concrete realization of the Mandelstam 'tHooft dual superconducting mechanism for confinement which can now be confronted by experiment.

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FIGURE CAPTIONS

Fig. 1 Dirac string connecting oppositely charged particles.

Fig. 2 Diagram representing string cancellation mechanism of eq. (2.13).

Fig. 3 The points represent the particles on the π , ρ , ω , and ϕ Regge trajectories, and the solid lines are the corresponding linear fits.

Dotted lines are Regge trajectories calculated from eqs. (5.14), (5.15) and (5.16):

$--$ $m_q = 100$ MeV, $\cdot \cdot \cdot$ $m_q = 300$ Mev.

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